ORBITAL DYNAMICS OF PARTICLE CAPTURE

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Abstract. In this paper we examine the transformations and observations required to compute the orbit of a dust particle captured in an aerogel by an Earth orbiting satellite.

Contents

Introduction

One way to study the structure of the universe is to study the dust particles which permeate the medium. However, studying dust particles in space from the ground, or even from an orbiting satellite, would be highly improbable using normal astronomical tools such as the telescope. Thus, the alternative is to capture the particles and then return them to the laboratory to be studied. Unfortunately, this process of capturing dust particles and then examining them on the ground, causes the loss of a potentially valuable piece of information; where did the dust grain originate? The only way to determine this information is to know the orbit the particle was in, but once the satellite returns with a foam block full of dust, the orbits will be hard to determine. In order to circumvent this problem then, it is necessary to record the orbit of the particles upon capture. Examining this process is the goal of this paper.

Knowing the orbit of the satellite, the positioning of the satellite in a chosen inertial frame at the time of impact, and the trajectory of the particle relative to the satellite frame, the orbit of the particle in the inertial frame can be computed. This process of collecting observations and ultimately computing the particle orbit is composed of three major procedures; repeatedly computing the orbit of the satellite, recording sufficient data at impact to position the satellite and particle, and the actual computation of the particle orbit. The reason the satellite orbit must be repeatedly computed is that despite the relative simplicity of $F = m * a$, closed form solutions do not exist for the the multi-body problem that the solar system represents. In fact closed form solutions exist only for the classic two-body problem with spherically distributed masses.[1]. Thus to avoid costly numerical integration, the satellite's orbit around the Earth is approximated as a two-body model and the perturbations arising from the oblateness of the Earth and the

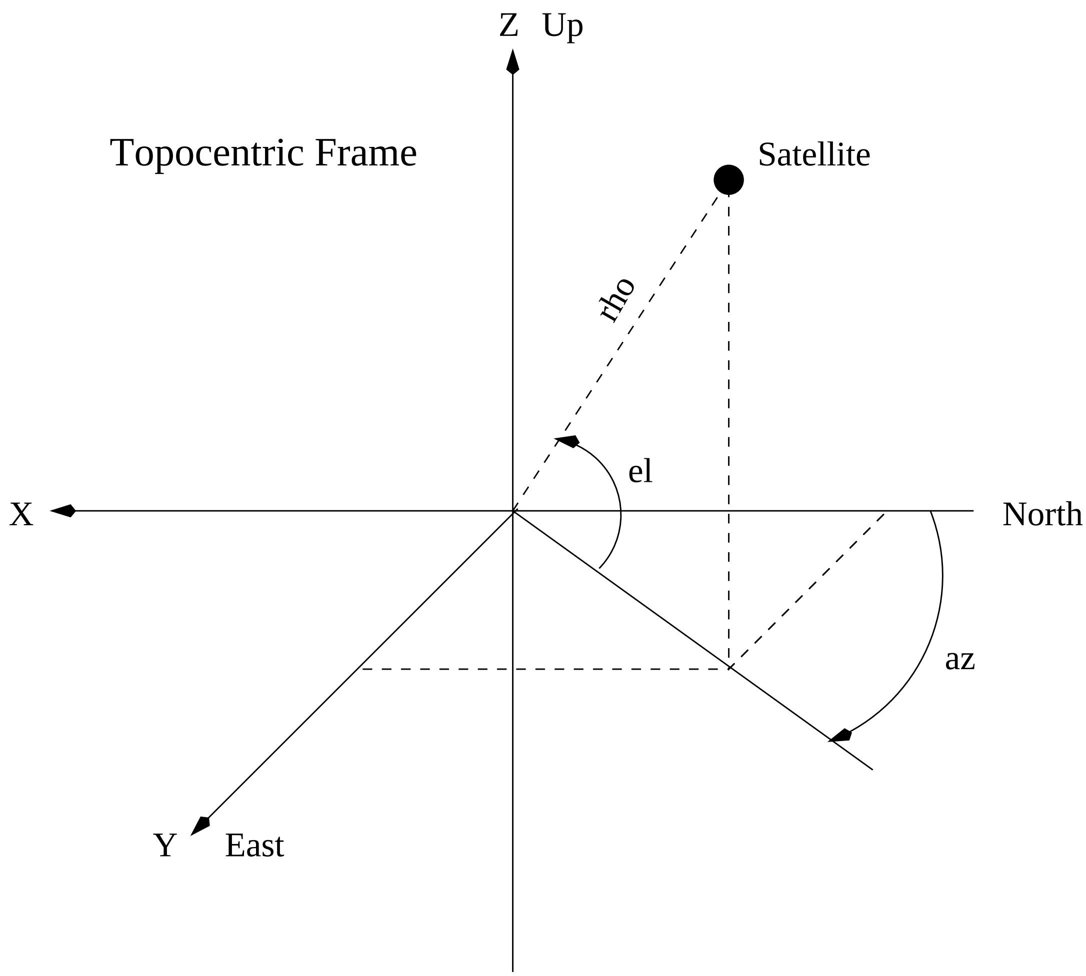
[1] Except in some simple or degenerate cases

presence of such bodies as the sun is accounted for by periodically recomputing the orbit of the satellite.

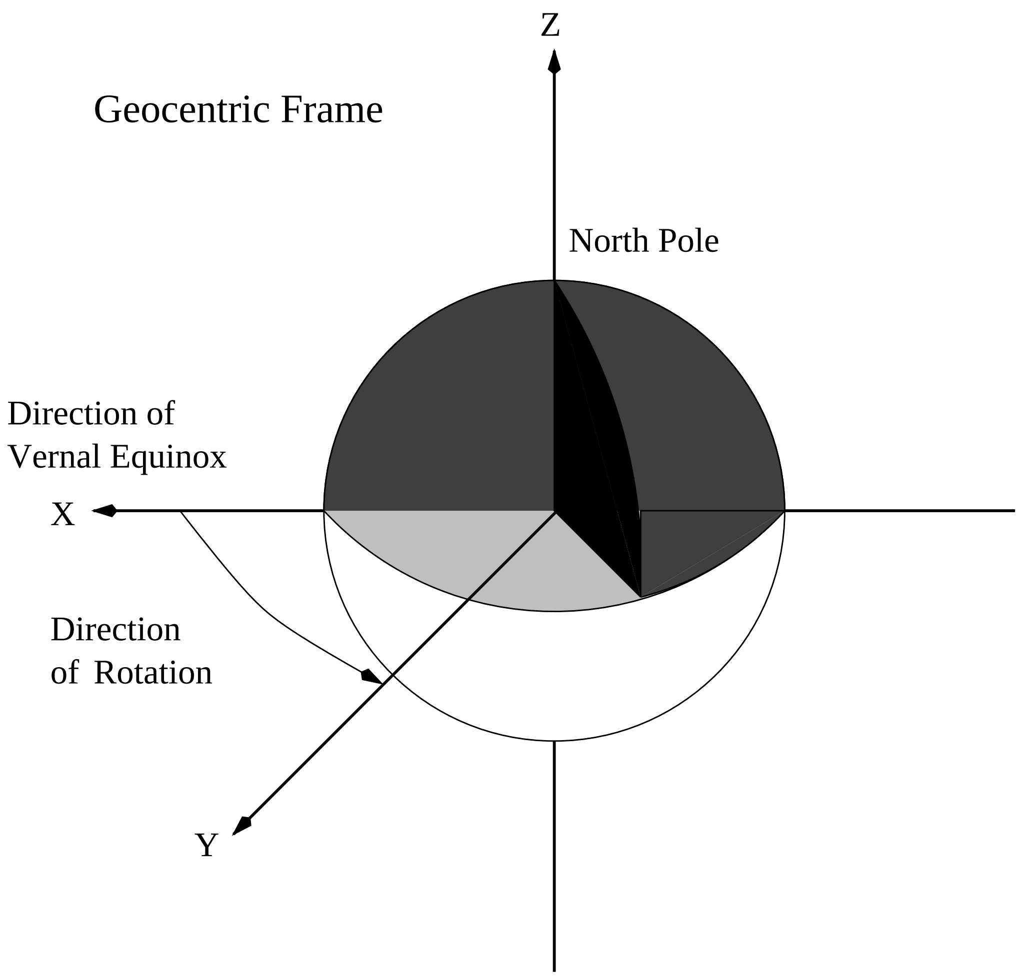
Notation. *Since there will be several variables representing velocities and position vectors in several different coordinates frames, the following notation will be used: A vector will be assumed to have three components, defined by the normals of the particular frame, and all vectors will be denoted by an over-arrow. Furthermore, each vector will have a left hand subscript denoting the coordinate frame of that vector, and a right hand subscript which will denote which object is represented by the vector. For example, the vector* ${}_G\overrightarrow{R}_s$ *would represent the position vector of a satellite in the Geocentric frame. The standard astronomical frames and their abbreviations for this paper are as follows:*

$$T : Topocentric$$
$$G : Geocentric$$
$$P : Perifocal$$
$$S : Satellite'sFrame(defined)$$

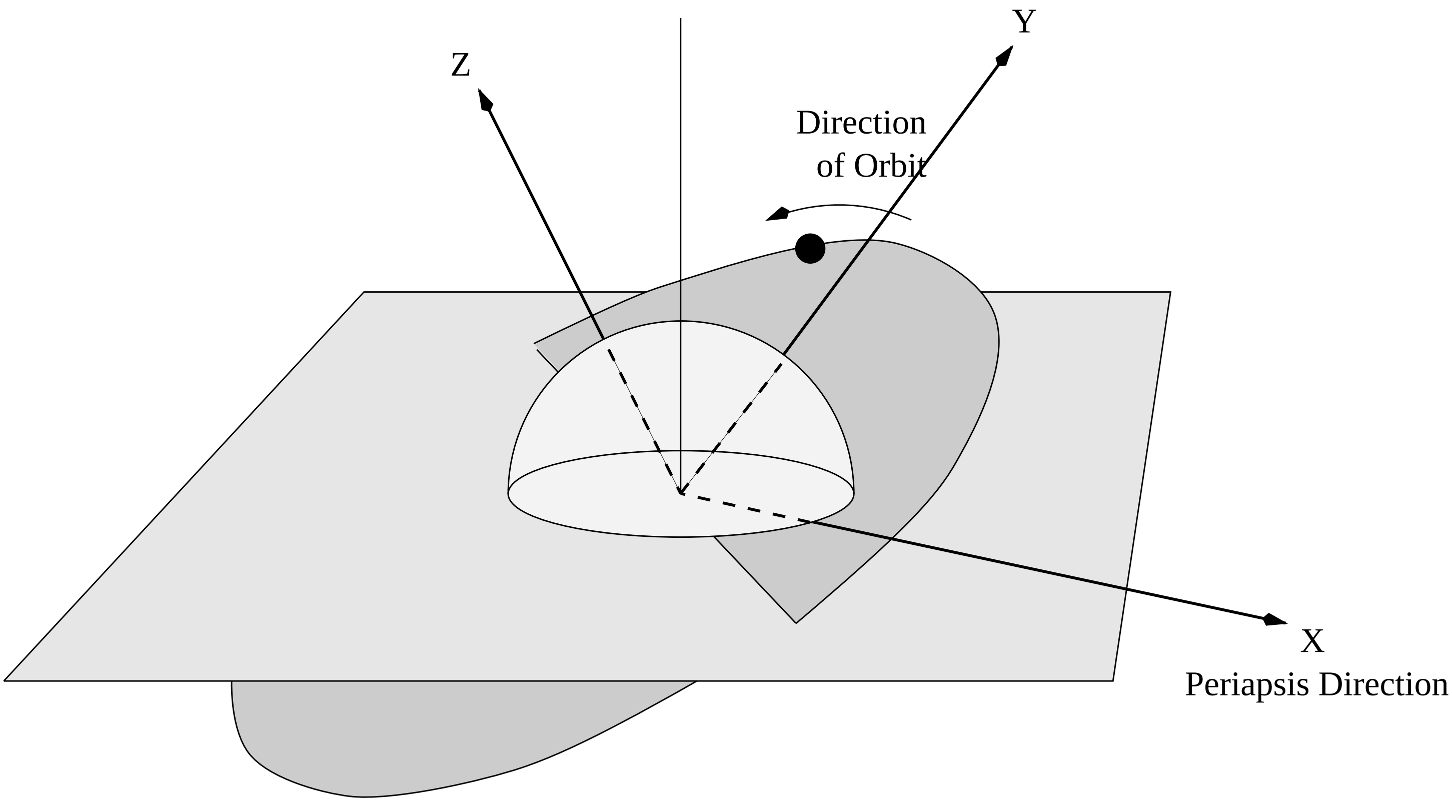
Definition. *The topocentric Frame is the "observatory" frame in that it utilizes the standard azimuth and elevation angles to describe the position of an object in space. The fundamental plane is the horizon with the Z axis pointing straight up, the X axis pointing south, and the Y axis pointing East. The azimuthal angle is the measure between the radial vector and the north axis and the elevation angle is the measure between the horizon, or the X-Y plane, and the radial vector. The third coordinate is* ρ *which is the radial distance from the origin to the point. The following figure represents the relationship between these coordinates.*



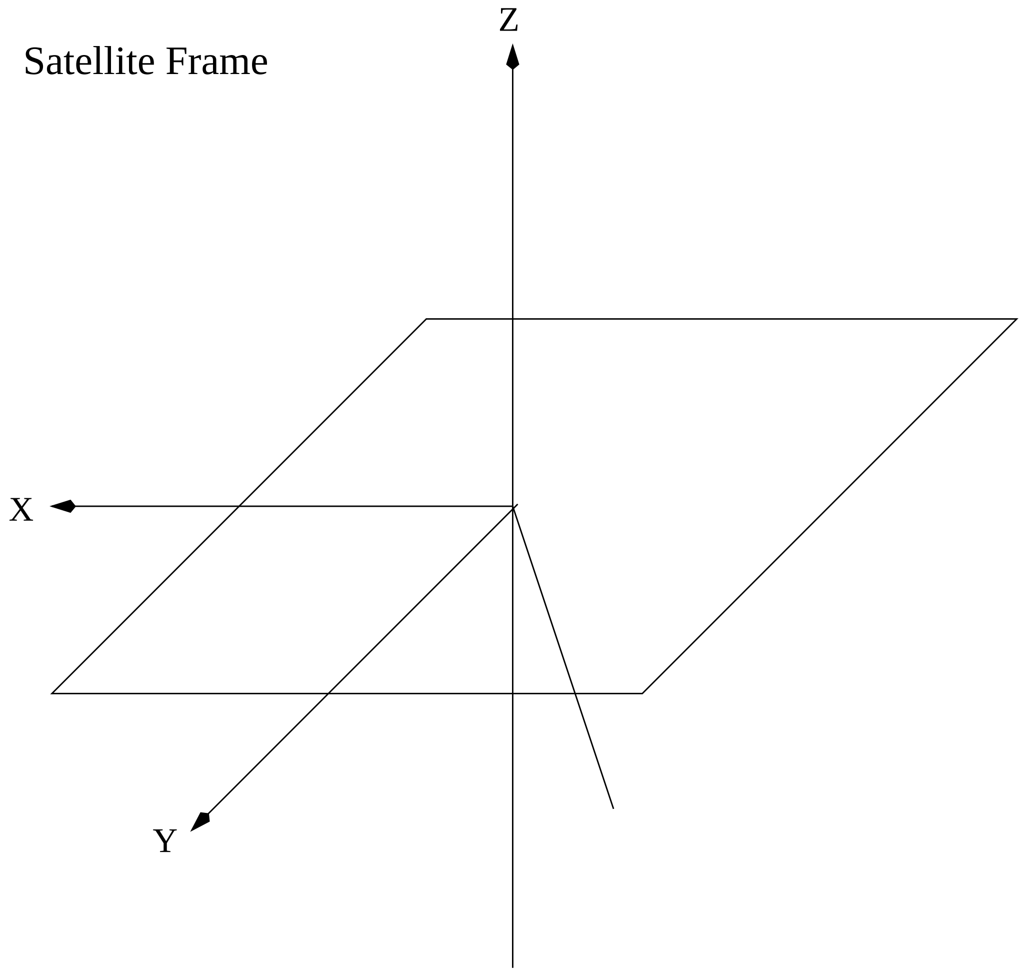
Definition. *In the Geocentric Frame the origin is the Earths center, the fundamental plane is the equator, the* Z *axis runs through the north pole, the* X *axis points in the celestial direction of the vernal equinox* τ*, and the* Y *axis is perpendicular to the* X *axis rotated in the direction of the Earths polar rotation. The direction of the vernal defined by a line originating from the sun and extending to the Earth at the time of the vernal equinox. This frame revolves with the Earth around the sun, but does not rotate with the earth; instead the earth rotates in the frame. Thus, transforming a topocentric frame vector into this frame removes the dependency of the Earths rotation from the vector.*



Definition. *The Perifocal Frame is the most convenient coordinate system for expressing the orbit of an object because the fundamental plane is the plane of the orbit itself. The X axis is in the direction of periapsis, the Y axis lies in the fundamental plane rotated 90 degrees from the X axis in the direction of the orbit, and the Z-axis lies along the cross product of X and Y. The direction of periapsis is the direction from the center of the Earth to the point of closest approach.*



Definition. *The satellite frame is assumed to be a right hand coordinate frame on the body of the satellite as defined in the drawing below. The fundamental plane is the capture surface and the positive Z direction is away from the center of the satellite. It is also assumed that the satellite is equipped with three gyroscopes, each of which is calibrated onto one of the Geocentric Frame coordinates so that the nine direction cosines between the Geocentric and the satellite frame are readily available.*



Determining the Orbit of the Satellite

The process of determining the orbit of the satellite requires measuring six variables. These six variables can be measured by any one of numerous ground based methods: radar tracking, three position vectors (the classic Gauss problem [G]) etc. The choice of method is a function of equipment available and accuracy required. For the purpose of this paper, the method used will be a combination of radar tracking and Doppler shift measurements to determine the satellite orbit. In practice, this method is not particularly accurate and would only be used to determine a rough orbit for the satellite so that better measurements could be made at a future time. However, this method will serve to demonstrate the way in which the classic orbital elements of the satellite can be determined from observations.

In this method, a radar site located at latitude λ, longitude θ, and elevation h uses a Doppler equipped antennae at time t to make two observations of the satellite, giving the elevation $el1$ and $el2$, the azimuth $az1$ and $az2$, and the distance $\rho1$ and $\rho2$. The two measurements are made over a short period of time δt. The two data pairs can be averaged over δt to approximate the rate of change of el, az, and ρ. Thus, letting a dot represent the time derivative, we have the following six coordinates on the satellite:

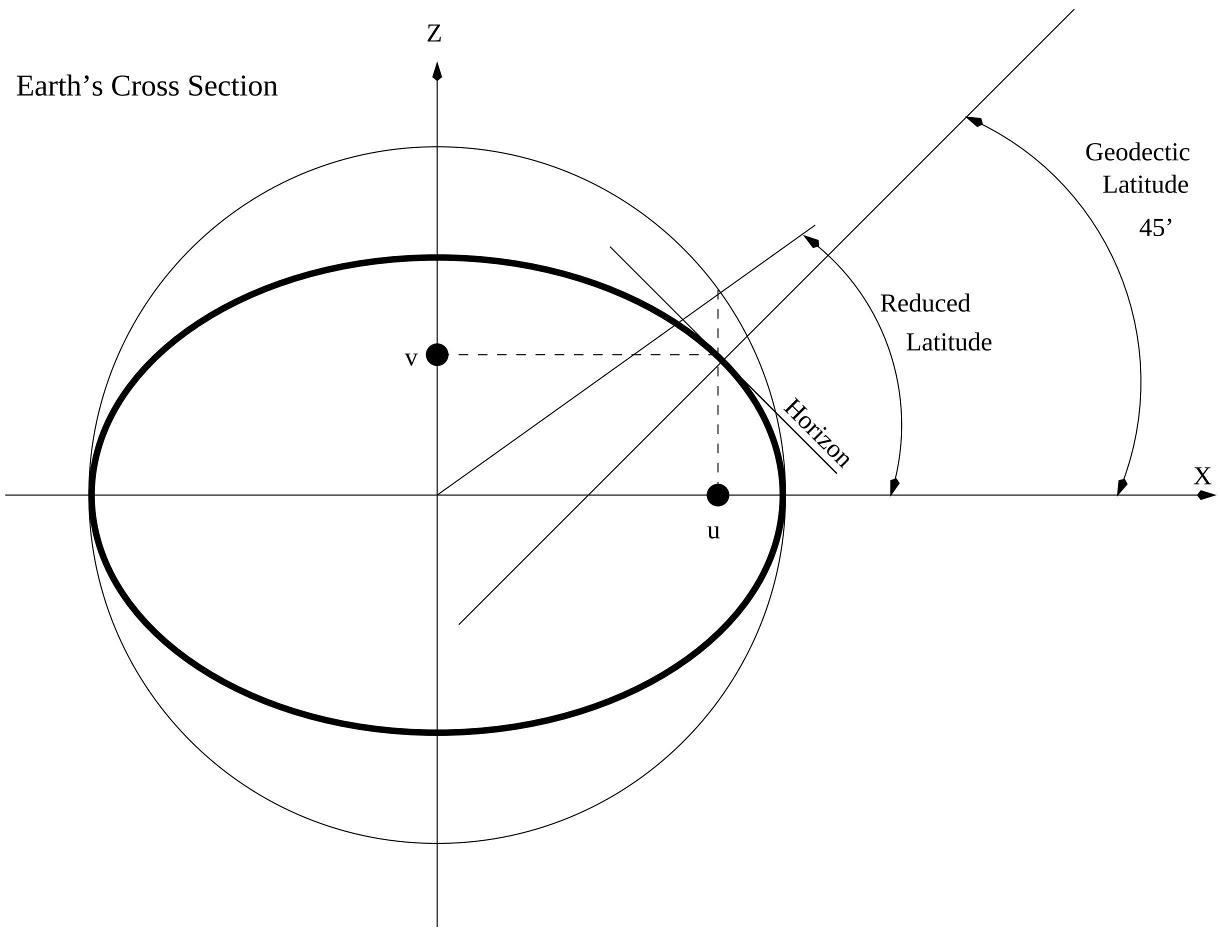
$$el = \frac{el2 + el1}{2}$$
$$az = \frac{az2 + az1}{2}$$
$$\rho = \frac{\rho 2 + \rho 1}{2}$$
$$\dot{el} = \frac{el2 - el1}{\delta t}$$
$$\dot{az} = \frac{az2 - az1}{\delta t}$$
$$\dot{\rho} = \frac{\rho 2 - \rho 1}{\delta t}$$

To transform these six variables into the topocentric frame requires a slightly modified spherical coordinate transformation as given in [BMW]:

$$_T\overrightarrow{R}_s = -\rho cos(el)cos(az)\hat{x} + \rho cos(el)sin(az)\hat{y} + \rho sin(el)\hat{z}$$
$$_T\overrightarrow{V}_s = A\hat{x} + B\hat{y} + C\hat{z}$$
$$A = -\dot{\rho} cos(el)cos(az) + \rho(\dot{el})sin(el)cos(az) + \rho(\dot{az})cos(el)sin(az)$$
$$B = \dot{\rho} cos(el)sin(az) - \rho(\dot{el})sin(el)sin(az) + \rho(\dot{az})cos(el)cos(az)$$
$$C = \dot{\rho} sin(el) + \rho(\dot{el})cos(el)$$

The next step is to convert the position and velocity vectors from the topocentric frame into the geocentric frame. The motivation for doing this is that the topocentric frame is far from being an inertial frame, in fact the whole frame is rotating with the Earth. The Geocentric Frame on the other hand, is centered at the earths core rather then the surface of the Earth and is basically fixed in space. These properties are both suitable for describing the orbits of low Earth orbiting satellites because, in the two body model, it is the interaction between the spacecraft and the center of the Earth which is relevant. Thus the coordinate transformation must remove the Earths rotation from the vectors as well as translate the origin - a process involving one translational component and three rotations. This process is unfortunately complicated by the fact that the Earth is not spherical; indeed the Earth is oblate, shaped rather like a pear. The significance of this emerges in the measurement of latitude of the observation sight. For example, if we use an ellipse as the model of the Earths cross section, which is more accurate than the circle, it is evident in the figure below that the location on the surface where the horizon is at an angle of 45 degrees to the equatorial plane is not halfway between the Earths equator and pole as it would be on the spherical model[2].

[2] This representation is largely exaggerated. The Earths eccentricity is roughly 0.08 which means that the point on the surface which is halfway between the north pole and the equator occurs at geodectic latitude of 45.09 degrees as opposed to 45.00 degrees. This is not negligible since on the scale of an orbiting satellite this discrepancy will amount to an error measurable in several kilometers



However, this is the angle given as the latitude of locations on the surface of the Earth. Thus, to accurately locate the observatory on the surface of the Earth it is necessary to consider the "station coordinates" u and v in the above diagram, as derived in [BMW], with the Earths equatorial radius a_e and eccentricity e as defined below:

$$u = \left| \frac{a_e}{\sqrt{1 - e^2 sin(\lambda)^2} + h} \right| cos(\lambda) \qquad a_e = 6378.145km$$

$$v = \left| \frac{a_e * (1 - e^2)}{\sqrt{1 - e^2 sin(\lambda)^2} + h} \right| sin(\lambda) \qquad e \ = 0.08182$$

Beside the above corrections for the latitude, it is also necessary to correct the longitude, θ, since a given longitude specifies different directions in space depending on the position of the Earths in its daily axial rotation. This correction is determined from the time of the observation t. However, as with the latitude corrections, there is a necessary consideration; in this case the consideration is the sidereal time. Sidereal time is the length of time required for the Earth to complete one rotation on its axis, so that the normal at the observation site would be pointing in the same direction in space. Since the Earth simultaneously revolves around the sun, the solar day is roughly four minutes longer than the time it takes for the Earth to rotate. Thus a sidereal day is approximately 23:56 long.

One method for computing the local sidereal time, θ_l is to look up the Greenwich Sidereal Time for a know date, call it θ_g at time t_0, and then use the following equation:

$$\theta_l = \theta_g + (t - t_0)\omega + \theta \qquad \omega = 7.292115856e^{-5}\frac{rad}{sec}\hat{z}$$

The local sidereal time can also be computed algorithmically from the time of day and the date [EB]. Using the local sidereal time, θ and the station coordinates of the observatory, u and v, the vector in the Geocentric Frame which points to the observatory, which is the origin of the topocentric Frame, can be computed by

$$\overrightarrow{L}_{T\mapsto G} = ucos(\theta)\hat{x} + usin(\theta)\hat{y} + v\hat{z}$$

The above vector defines the linear translation of the two frames. In order to complete the transformation of the topocentric vectors, it is necessary to compute the matrix which represents the rotation of the topocentric Frame relative to the Geocentric Frame. Using the latitude and local sidereal time this rotation matrix is as follows:

$$D_{T\mapsto G} = \begin{bmatrix} sin(\lambda)cos(\theta_l) & -sin(\theta_l) & cos(\lambda)cos(\theta_l) \\ sin(\lambda)sin(\theta_l) & cos(\theta_l) & cos(\lambda)sin(\theta_l) \\ -cos(\lambda) & 0 & sin(\lambda) \end{bmatrix}$$

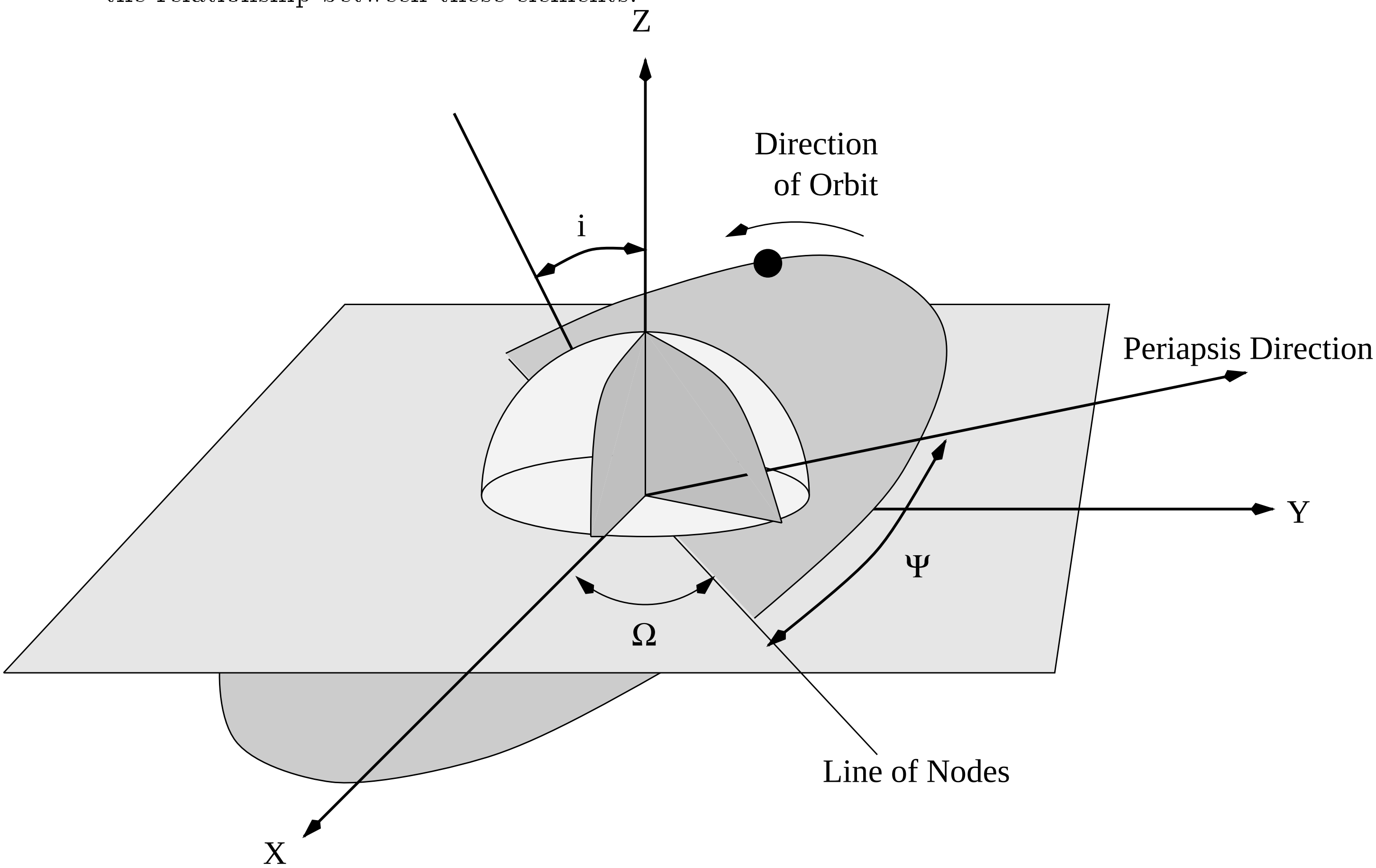
The process for computing the Geocentric position vector is to compute the vector sum of ${}_T\overrightarrow{R}_s$ and $\overrightarrow{L}_{T\mapsto G}$ and then apply the rotation $D_{T\mapsto G}$. The Geocentric velocity vector is computed by rotating the topocentric velocity by $D_{T\mapsto G}$ and then applying a correcting term of the form $\omega \times R$ to compensate for the relative rotation of the two frames:

$$_G\overrightarrow{R}_s = D_{T\mapsto G}(\overrightarrow{L}_{T\mapsto G} +_T \overrightarrow{R}_s)$$
$$_G\overrightarrow{V}_s = D_{T\mapsto G}(_T\overrightarrow{V}_s) + \omega \times_G \overrightarrow{R}_s$$

The task of determining the orbit of the satellite with respect to an inertial frame is complete. However, during the satellites orbit about the Earth, all six of the variables in the two vectors ${}_GR_s$ and ${}_GV_s$ will change as a function of time. A generalized method for determining these future values of R and V does exist [BMW], but it is very complicated and unnecessary since the orbit of the satellite in this model is very well behaved; well behaved in that the satellite is assumed to be orbiting elliptically and at some inclination to the equatorial plane. One way to capitalize on these assumptions is to describe the orbit of the satellite specially in terms of the shape of the orbit. For example, to determine the shape of the ellipse requires knowing only the semi-major axis a and the eccentricity e[3] Now if the position and orientation of this ellipse are given then the orbit of the satellite would be determined. The method to describe the position of the ellipse is to find the inclination i, the longitude of the ascending node Ω, and the argument of periapsis Ψ.

[3]Other combinations of variables are also possible sine relationships exist among them, for example $a^2 + b^2 = c^2$ $\frac{e=c}{a}$

The inclination i is angle between the plane of the orbit and the fundamental plane of the inertial frame which is the Earths equatorial plane in this case. The longitude of the ascending node, Ω, is the angle, in the fundamental, plane between the geocentric x-axis and the line formed by the intersection of the orbital and fundamental planes where the satellite is moving with in the northernly direction. The argument of periapsis, Ψ, is the angle in the orbital plane, between the ascending node and the line of periapsis, measured in the direction of the satellites orbit. This set of five variables which define the shape and orientation of a conic section orbit in space are referred to as the Classical Elements and completely determine a unique orbit in space except for the position of the satellite in the orbit. To determine where the satellite is requires a sixth element, called the true anomaly, ν which is the angle between the line of periapsis and the satellite's position, measured in the satellites direction of motion. Since this is the only variable which determines the satellites position, the other five variables remain constant; except for perturbations outside the two body approximation. The following figure, adapted from [BMW] depicts the relationship between these elements.



Following the derivation given in [BMW], the first step is to compute two cross products:

$$\overrightarrow{H} = \overrightarrow{R} \times \overrightarrow{V}$$

$$\overrightarrow{N} = \hat{z} \times \overrightarrow{H}$$

The first vector, $\overrightarrow{H}$, is perpendicular to both the velocity and position vectors and is thus normal to the orbital plane. In fact, $\overrightarrow{H}$ is precisely the angular momentum vector and its magnitude is the magnitude of the angular momentum of the satellite. The vector $\overrightarrow{N}$, on the other hand, is perpendicular to the $\hat{z}$ axis and the the vector $\overrightarrow{H}$ and thus must lie in both the equatorial plane of the geocentric frame and the orbital plane thus representing the intersection of the two planes, called the line of nodes. The magnitude of $\overrightarrow{N}$ is of no importance. The next step is to form the eccentricity vector:

$$\mu = GM = 3.986012 * 10^5 \frac{km^3}{sec^2}$$

$$\overrightarrow{E} = \frac{1}{\mu}\left[\left(\|\overrightarrow{v}\|^2 - \frac{\mu}{\|\overrightarrow{r}\|}\right)\overrightarrow{r} - \left(\overrightarrow{r}\cdot\overrightarrow{v}\right)\overrightarrow{v}\right]$$

The eccentricity vector is useful because its norm $\|\overrightarrow{E}\|$ is equal to the eccentricity of the satellites orbit, and the vector points in the direction of perigee, which is the direction of periapsis or closest approach. These three vectors can be combined to determine the six classical elements as follows:

$$e = \|\overrightarrow{e}\|$$

$$p = \frac{\|\overrightarrow{H}\|^2}{\mu} = a(1-e^2)$$

$$i = cos^{-1}\left(\frac{\overrightarrow{H}\cdot\hat{z}}{\|\overrightarrow{H}\|}\right)$$

$$\Omega = cos^{-1}\left(\frac{\overrightarrow{N}\cdot\hat{x}}{\|\overrightarrow{N}\|}\right) \qquad \text{if}\overrightarrow{N}\cdot\hat{y} > 0 \implies \Omega < \Pi$$

$$\Psi = cos^{-1}\left(\frac{\overrightarrow{N}\cdot\overrightarrow{E}}{\|\overrightarrow{N}\|\|\overrightarrow{E}\|}\right) \qquad \text{if}\overrightarrow{E}\cdot\hat{z} > 0 \implies \Psi < \Pi$$

$$\nu_0 = cos^{-1}\left(\frac{\overrightarrow{E}\cdot\overrightarrow{R}}{\|\overrightarrow{E}\|\|\overrightarrow{R}\|}\right) \qquad \text{if}\overrightarrow{R}\cdot\overrightarrow{V} > 0 \implies \nu < \Pi$$

Notice that the previously mentioned element a can be calculated by the equation given above for the semi-latus rectum p. Also, it is important to understand that these equation apply to an elliptical orbit. For example, if the orbit was circular, then there would be no perigee and Ψ would be undefined. There are different classical elements available to handle these types of orbits and the reader is referred to [BMW] for further discussion.

There is one possible degenerate case for the current model since if the orbit is purely in the equatorial plane, i.e. $i = 0$, then the line of nodes will not exist and both Ω and Ψ will be undefined. In this situation, two new angles are used, Π and l_0. Π is the angle between the $\hat{x}$ axis and perigee, measured in the direction of orbital flight. Then, $l_0 = \Pi + \nu_0$ The orbit of the satellite is now completely determined.

Determining the Position of the Satellite at Time of Particle Impact

Now that the orbit of the satellite has been determined, the next step in determining the orbit of a captured particle is to locate the position of the satellite at the time of capture. The method for doing this it to use a time-of-flight equation to compute the anomaly of the satellite at the time of capture and then transform the classical elements back into the geocentric frame. Assume that a particle capture occurs at time t and the anomaly ν_0 from the last section was computed at time t_0. The number of times the satellite has completed a full orbit since the first positioning can be calculated by using Kepler's equation for the period of an elliptical orbit as follows:

$$\frac{k = t - t_0}{\frac{2*\Pi}{\sqrt{\mu}} a^{3/2}}$$

Following [BMW], the intermediate values η_0 and M are:

$$\eta_0 = cos^{-1}\left(\frac{e + cos(\nu_0)}{1 + ecos(\nu_0)}\right)$$
$$M = \sqrt{\frac{\mu}{a^3}}(t - t_0) - 2k\Pi + \eta_0 - esin(\eta_0)$$

The next step is to compute the eccentric anomaly η from the equation: $M = \eta - esin(\eta)$. Unfortunately, this is a transcendental equation which does not have a closed form solution. One way to find η is to use Newton's method. Let superscripts represent iterations with $\eta^0 = \Pi$. Then Newton's method would proceed as:

$$M^n = \eta^n - esin(\eta^n)$$
$$\eta^{n+1} = \eta^n + \frac{M - M^n}{1 - ecos(\eta^n)}$$

The above equation, with starting value $\eta^0 = \Pi$ should converge rapidly for all values of e even if close to one (near parabolic). Once η is determined, then ν can be found using:

$$\nu = cos^{-1}\left(\frac{e - cos(\eta)}{ecos(\eta) - 1}\right)$$

The position of the satellite is now known in the celestial elements. At this point, there are several methods for determining the vectors ${}_GR_s$ and ${}_GV_s$. Since the satellites orbit is a well behaved elliptical orbit, this model will utilize a two stage coordinate transformation, going first to the perifocal system using:

$$r = \frac{p}{1 + ecos(\nu)}$$
$$_P\overrightarrow{R}_s = rcos(\nu)\hat{x} + rsin(\nu)\hat{y}$$
$$_P\overrightarrow{V}_s = \sqrt{\frac{\mu}{p}}\left[-sin(\nu)\hat{x} + (e + cos(\nu))\hat{y}\right]$$

These perifocal coordinates can now be transformed into the geocentric frame using the rotation matrix:

$$D_{P\mapsto G} = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix}$$

$$R_{11} = cos(\Omega)cos(\Psi) - sin(\Omega)sin(\Psi)cos(i)$$
$$R_{12} = -cos(\Omega)sin(\Psi) - sin(\Omega)cos(\Psi)(cos(i)$$
$$R_{13} = sin(\Omega)sin(i)$$
$$R_{21} = sin(\Omega)cos(\Psi) + cos(\Omega)sin(\Psi)cos(i)$$
$$R_{22} = -sin(\Omega)sin(\Psi) + cos(\Omega)cos(\Psi)cos(i)$$
$$R_{23} = -cos(\Omega)sin(i)$$
$$R_{31} = sin(\Psi)sin(i)$$
$$R_{32} = cos(\Psi)sin(i)$$
$$R_{33} = cos(i)$$

Then, the transformation into the Geocentric frame is as follows:

$$_G\overrightarrow{R}_s = D_{P\mapsto G}(_P\overrightarrow{R}_s)$$
$$_G\overrightarrow{V}_s = D_{P\mapsto G}(_P\overrightarrow{V}_s)$$

Now that the satellite can be positioned at the time of impact, t, it only remains to determine the direction the satellite is facing and then the particle's orbit. Knowing which orientation of the satellite's capture surface at any given instance is not something which can be easily measured from ground observations. Thus, the satellite must itself have instrumentation capable of measuring an orientation. The simplest way to do this is to equip the satellite with a gyroscope, which basically measures the direction cosines between a fixed direction in space and a fixed axis on the satellite. Reviewing classical dynamics, [MT], six relationships exist between the nine direction cosines of two co-origin rotated frames. This would imply that only three variables, and thus one gyroscope, need provide direction cosines. However, the relationships between the direction cosines are both non-linear and dependent. Consequently, the system can be determined with only three angles, but the three angles need to be on direction cosine from each axis. Since this would require three gyroscopes, it is no simplification to assume that only three direction cosines are available. Thus, for the purpose of this paper, the satellite will be equipped with three gyroscopes which will provide the nine direction cosines between the satellite frame and the Geocentric frame. With these direction cosines, the following rotation matrix applies:

$$D_{S\mapsto G} = \begin{bmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} \\ \lambda_{21} & \lambda_{22} & \lambda_{23} \\ \lambda_{31} & \lambda_{32} & \lambda_{33} \end{bmatrix}$$

The assumptions made are that the particle is moving much faster than the satellite is rotating, and the position vector to the satellite is the same as the position vector to the particle. The first assumption is reasonable since most of these particles will be moving at hyper-velocities and the second is reasonable because the scale of the satellite is in meters, but the position vector will be a few hundred kilometers.

At the time of impact, the particle is captured non-destructively in an aerogel. To determine the orbit of the particle requires knowing how fast the particle was moving relative to the satellite and in what direction. Whether this is done with sensors on the satellite or by analyzing the burn track in the aerogel once it has returned to Earth, the sufficient data would be the magnitude of the velocity and two spherical angles measured relative to the satellite's frame, which point in the direction of entry. Let $\|v\|$ represent the magnitude of the velocity and β_1 and β_2 represent the spherical angles. This data can be transformed into vectors simply by writing out the spherical coordinate representation as follows:

$${}_S\overrightarrow{V}_p = \|v\|(sin(\beta_1)cos(\beta_2)\hat{x} + sin(\beta_1)sin(\beta_2)\hat{y} + cos(\beta_1)\hat{z}$$

Transforming the position and velocity vectors into the geocentric frame is simply a matter of rotating the velocity vector and adding it to the satellites velocity:

$${}_G\overrightarrow{R}_p =_G \overrightarrow{R}_s$$
$${}_G\overrightarrow{V}_p =_G \overrightarrow{V}_s + D_{S\mapsto G}({}_S\overrightarrow{V}_p)$$

At this point, the orbit of the particle is determined.

Road-map and C-Code for Computing Particle Orbit

The following is a road-map intended to graphically illustrate the relationship between the transformations discussed in this paper and also simplify following the logic. The map is constructed of three main features: circles represent data colloection points and the input data and its origin is specified. Arrow indicate the flow of data from one stage of the transformation to the next. Boxes are intended to operate like black boxes in that each represents a transformation of data with inputs and output. Each of these transformation black-boxes has a corresponding function written in MicroSoft QuickC in the following section, which performs the transformation as layed out by the black-box. Some of the algorithms are adapted from [EB].

**** This portion still unfinished ****

References

[BMW] Roger Bate, Donald Mueller, Jerry White, *Fundamentals of Astrodynamics*, 1971.

[MT] Jerry Marion, Stephen Thornton, *Classical Dynamics*, 1988, pp. 1-9.

[G] Carl Gauss (Charles Davis), *Theory of the Motion of the Heavenly Bodies Revolving about the Sun in Conic Sections*, A translation of "Theoria Motus" (1857), 1963.

[EB] Eric Burgess, *Celestial Basic*, 1982.

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